

Matter and twin matter in bimetric MOND

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ABSTRACT

Bimetric MOND (BIMOND) theories, propounded recently, predict peculiar gravitational interactions between matter and twin matter (TM). Twin matter is the hypothetical matter that might couple directly only to the second metric of the theory, as standard matter couples only to the first. Considerations of cosmology in the BIMOND framework suggest that such TM might exist and copy matter in its attributes. Here I investigate the indirect interactions that BIMOND theories predict between local, non-relativistic mass concentrations of matter and TM. The most salient result is that in the deep-MOND regime of the matter-TM-symmetric theories, TM behaves as if it has a negative gravitational mass relative to matter (active and passive, with the inertial mass still positive). To wit, interaction within each sector is attractive MOND gravity, but between matter and TM it is repulsive MOND gravity. Using the space-conformal invariance of the theory in the deep-MOND regime, I derive various exact results; e.g., the repulsive force between a matter and TM point masses (space-conformal theories are a natural framework for masses of opposite signs). In the high-acceleration regime, the interaction depends on a parameter, β (the strength of the Einstein-Hilbert action for matter). For the favored value $\beta = 1$, matter and TM do not interact in this regime; for $\beta < 1$ they attract; and for $\beta > 1$ they repel each other. Such interactions may have substantial ramifications for all aspects of structure formation, such as matter distribution, peculiar velocities, and effects on the CMB. The repulsive interactions probably lead to segregation of matter and TM structures, leading, in turn, to intermeshing of the respective cosmic webs, with high-density nodes of one sector residing in the voids of the other (possibly conducing to efficient evacuation of the voids). Weak gravitational lensing by TM seems the best way to detect it or constrain its attributes. In the MOND regime a TM body acts on matter photons as a diverging lens. Strong lensing occurs in the high acceleration regime, and thus depends on β . For $\beta = 1$, a TM mass does not bend (matter) light in the high-acceleration regime: no strong lensing effects of TM are expected in this case. I also discuss briefly asymmetric theories.

Subject headings:

1. Introduction

Bimetric theories of gravity involve two metrics as independent degrees of freedom: one felt directly by standard matter, $g_{\mu\nu}$, and an auxiliary metric, $\hat{g}_{\mu\nu}$. Aspects of such theories have been extensively discussed; e.g., by Isham, Salam, & Strathdee (1971), by Rosen (1974), and, for some recent treatments, with references to other and to earlier work, see, e.g., Boulanger & al. (2001), Damour & Kogan (2002), Blas, Deffayet, & Garriga (2006), Bañados, Ferreira, & Skordis (2009), and Bañados, & al. (2009). It may be the case that the metric $\hat{g}_{\mu\nu}$ is indeed merely an auxiliary field in the description of gravity of standard matter. It is, however, natural to suppose that $\hat{g}_{\mu\nu}$ comes with a matter sector of its own, and should be viewed on a par with $g_{\mu\nu}$. This opens the way for introducing twin matter (TM), which may couple to its ilk in the same way as standard matter does (electromagnetically, weakly, etc.), but which couples gravitationally only to $\hat{g}_{\mu\nu}$, just as matter couples only to $g_{\mu\nu}$. Since the two metrics are coupled through a gravitational term in the action, matter and TM do couple indirectly by some unconventional gravitational interaction.

A bimetric formulation of MOND (BIMOND) has been propounded recently (Milgrom 2009b), which points even more forcibly to the possible existence of TM.

BIMOND is governed by an action of the form

$$I = -\frac{1}{16\pi G} \int [\beta g^{1/2} R + \alpha \hat{g}^{1/2} \hat{R} - 2(g\hat{g})^{1/4} f(\kappa) a_0^2 \mathcal{M}(\bar{\Upsilon}/a_0^2)] d^4x + I_M(g_{\mu\nu}, \psi_i) + \hat{I}_M(\hat{g}_{\mu\nu}, \chi_i). \quad (1)$$

Here $\bar{\Upsilon}$ represents a collection of scalar variables formed by contractions of the acceleration-like tensors $C_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha - \hat{\Gamma}_{\beta\gamma}^\alpha$, where $\Gamma_{\beta\gamma}^\alpha$ and $\hat{\Gamma}_{\beta\gamma}^\alpha$ are the Levi-Civita connections of the two metrics. Also, $\kappa \equiv (g/\hat{g})^{1/4}$ (g and \hat{g} are minus the determinants of the two metrics), G is the phenomenological Newton constant, and I use unites where $c = 1$. (\mathcal{M} may also depend on scalars constructed from the two metrics, such as κ .) I_M is the action for matter, whose degrees of freedom and their derivatives are collectively marked by ψ_i ; these interact among themselves, and couple to the metric $g_{\mu\nu}$ alone in the standard way, and similarly \hat{I}_M is the TM action. I have worked, in particular, with favorite choices of a the scalar variable in \mathcal{M} built from the tensor

$$\Upsilon_{\mu\nu} = C_{\mu\lambda}^\gamma C_{\nu\gamma}^\lambda - C_{\mu\nu}^\gamma C_{\lambda\gamma}^\lambda; \quad (2)$$

for example,

$$\bar{\Upsilon} = -\frac{1}{2} g^{\mu\nu} \Upsilon_{\mu\nu}. \quad (3)$$

To obviate possible confusion, especially in the present MOND context, I note at the outset that the TM is not the putative dark matter (DM), and is not taken to play its role of enhancing gravity in galactic systems. It is still MOND departure from standard general relativity (GR) that replaces dark matter. TM may still produce some effects, e.g., in structure formation, that are conventionally attributed to cosmological DM. It may also linger in otherwise matter-dominated territory to produce some visible effects (see 4.0.3).

We have no a priori idea of how the TM sector looks like; neither do we have any observational information on the subject. Does it exist at all? Is it made of the same ‘stuff’ as matter, and is it subject to the same physics? Is it present in the Universe in the same amounts? Has it undergone similar processes in cosmic history (big bang, inflation, seed fluctuations, structure formation, etc.)? We do not even know whether matter coupling to gravity is the same in the two sectors. To be able to progress I shall assume symmetry between the sectors to the effect that TM duplicates matter in its composition, interactions, etc., and that if $(\hat{g}_{\mu\nu}, \chi_i)$ is the same configuration as $(g_{\mu\nu}, \psi_i)$, then $I_M(g_{\mu\nu}, \psi_i) = (\alpha/\beta)\hat{I}_M(\hat{g}_{\mu\nu}, \chi_i)$. This ensures that in the absence of the interaction \mathcal{M} , the dynamics within the two sectors are identical. With the interaction the above assumptions do not ensure complete symmetry when $\alpha \neq \beta$.

In cosmology, the above assumptions plus the assumptions that the cosmic matter contents of the two sectors are the same, and a symmetric choice of $f(\kappa) = (\beta\kappa + \alpha\kappa^{-1})/(\alpha + \beta)$, was shown (Milgrom 2009b) to lead to cosmologies in which $\hat{g}_{\mu\nu} = g_{\mu\nu}$, with each metric describing a standard, Friedmann-Robertson-Walker (FRW) cosmology, with a cosmological constant $\Lambda = -a_0^2\mathcal{M}(0)/(\alpha + \beta)$. This automatically retains the known successes of such a cosmology. Cosmology is thus, arguably, the strongest motivation for postulating that TM exists. In such BIMOND cosmologies, G/β appears in the matter sector of such a cosmology as Newton’s constant. This may induce us to prefer $\beta \approx 1$. The value $\beta = 1$ is also special because such BIMOND theories have a simple limit when $a_0 \rightarrow 0$: they tend to GR in the matter sector when $\mathcal{M}'(z) \rightarrow 0$ for $z \rightarrow \infty$, which is required for the nonrelativistic (NR) limit to yield Newtonian dynamics for $a_0 \rightarrow 0$. I have not yet investigated the limit $a_0 \rightarrow 0$ for BIMOND theories with $\beta \neq 1$ to see to what extent they differ from GR. I will, none the less, keep the discussion more general, taking $\beta = 1$, as an example, only at a later point.

While on cosmological average the two metrics might be the same, departures from equality must occur due to random fluctuations in density, which are different in the two sectors. To treat large-scale-structure formation through the development of small perturbations on the background of the expanding Universe (including their imprints on the CMB, etc.) we need to expand the BIMOND equations of motion around the double FRW metric describing cosmology at large.

Note, importantly in this context, that even if early on the Universe is characterized by high accelerations (as measured, e.g., by cH , where H is the Hubble parameter), what determines whether we are in the MOND regime is the argument of the interaction function \mathcal{M} . This argument is small when the two metrics are near each other. Thus, structure formation in the initial stages, in an otherwise symmetric universe, occurs in the MOND regime. So, even at small amplitude of fluctuations the development is nonlinear, as the MOND potentials are not linear in the density fluctuations, right from the outset. Structure formation, in all its aspects, such as the distribution of matter, peculiar motions, and CMB fluctuations, are thus phenomena that might be greatly affected by the new physics inherent in BIMOND, in particular by the possible effects of TM. The treatment of this problem has to be left to numerical simulations.

In this paper I take up the more modest task of investigating the dynamics of well formed systems involving well separated masses of matter and TM, of sizes much smaller than cosmic scales. These are then taken to be NR systems on a double Minkowski background ($\hat{g}_{\mu\nu} \approx g_{\mu\nu} \approx \eta_{\mu\nu}$). The appropriate limit of BIMOND theories under these circumstances was shown to be the class of NR MOND theories described in detail in Milgrom (2009c).

The analysis here includes derivation of the forces between combinations of matter and TM bodies, and also the characteristics of gravitational lensing of matter photons by TM.

In section 2, I describe the field equations that govern NR interactions within and between the two matter sectors. In section 3, I specialize to the class of theories that are fully symmetric in matter and TM. In section 4, I consider some possible observational consequences of the existence of TM and its peculiar interaction with matter, including gravitational lensing. In section 5, I look briefly at examples of asymmetric theories. Section 6 is a discussion.

2. Formalism

For a system of slowly moving matter and TM distributions ρ and $\hat{\rho}$, respectively, on a double Minkowski background, for the choice of variable $\tilde{\Upsilon}$ as in eq.(3), the solution for the metrics of the BIMOND field equations is (in some gauge)

$$g_{\mu\nu} = \eta_{\mu\nu} - 2\phi\delta_{\mu\nu}, \quad \hat{g}_{\mu\nu} = \eta_{\mu\nu} - 2\hat{\phi}\delta_{\mu\nu} \quad (4)$$

(Milgrom 2009b).¹ Here, the potentials ϕ and $\hat{\phi}$ are solutions of the NR limit of the BIMOND equations; these equations can be derived from the Lagrangian $L = \int \mathcal{L} d^3r$, with

$$\mathcal{L} = -\frac{1}{8\pi G} \{ \alpha(\vec{\nabla}\hat{\phi})^2 + \beta(\vec{\nabla}\phi)^2 - a_0^2 \mathcal{M}[(\vec{\nabla}\phi - \vec{\nabla}\hat{\phi})^2/a_0^2] \} + \rho(\frac{1}{2}\mathbf{v}^2 - \phi) + \gamma\hat{\rho}(\frac{1}{2}\hat{\mathbf{v}}^2 - \hat{\phi}). \quad (5)$$

The last two terms in the Lagrangian density tell us that ϕ is the (MOND) gravitational potential for matter, and $\hat{\phi}$ is that for TM, in the sense that the acceleration of matter and TM test particles is given by $\dot{\mathbf{v}} = -\vec{\nabla}\phi$, and $\dot{\hat{\mathbf{v}}} = -\vec{\nabla}\hat{\phi}$, respectively.²

Invariance of the theory to translations implies that for a closed system we have a vanishing total force:

$$\mathbf{F} = - \int (\rho\vec{\nabla}\phi + \gamma\hat{\rho}\vec{\nabla}\hat{\phi}) d^3r = 0, \quad (6)$$

¹Choices of the acceleration scalar arguments of \mathcal{M} other than that given in eq.(3) result in other NR limits of the corresponding BIMOND theory than those given by eq.(4). In general, more potentials are needed to describe each metric, and a more complicated set of coupled equations for these have to be solved (Milgrom 2009b).

²This NR theory is, in fact, the NR limit of a group of variations on the relativistic theory, where we can use in the contraction in $\tilde{\Upsilon}$, $\hat{g}^{\mu\nu}$ instead of $g^{\mu\nu}$, or use combinations of such variables, for more symmetry between the two metrics.

and the conserved momentum is

$$\mathbf{P} = \int (\rho \mathbf{v} + \gamma \hat{\rho} \hat{\mathbf{v}}) d^3 r. \quad (7)$$

I define $\hat{\rho}$ as nonnegative, so the sign of γ matters. In the spirit of what I said above about the relativistic theory, I take from now on $\gamma = \alpha/\beta$.³

The field equations are

$$\begin{aligned} \Delta \phi &= 4\pi G \beta^{-1} \rho + \beta^{-1} \vec{\nabla} \cdot (\mathcal{M}' \vec{\nabla} \phi^*) \equiv 4\pi G (\rho + \rho_p), \\ \Delta \hat{\phi} &= 4\pi G \beta^{-1} \hat{\rho} - \alpha^{-1} \vec{\nabla} \cdot (\mathcal{M}' \vec{\nabla} \phi^*) \equiv 4\pi G (\hat{\rho} + \hat{\rho}_p), \end{aligned} \quad (8)$$

where $\phi^* = \phi - \hat{\phi}$, and ρ_p and $\hat{\rho}_p$ playing the role of ‘phantom matter’ (PM) densities for the two sectors. ($\alpha = 0$, or $\beta = 0$ do not give MOND theories, and I exclude such values.) Note that for a given configuration the amounts of PM felt by matter and TM are, in general, different.

Subtracting one equation from the other we get a decoupled equation for ϕ^*

$$\vec{\nabla} \cdot [\mu^* (|\vec{\nabla} \phi^*|/a_0) \vec{\nabla} \phi^*] = 4\pi G (\rho - \hat{\rho}), \quad (9)$$

where

$$\mu^*(x) \equiv \beta - \frac{\alpha + \beta}{\alpha} \mathcal{M}'(x^2). \quad (10)$$

There is an equivalent, but more transparent, way to write the theory for the case $\alpha + \beta \neq 0$: Define

$$\tilde{\mathcal{M}}(z) = -\mathcal{M}(z/q) + \frac{\alpha\beta z}{(\alpha + \beta)q}, \quad (11)$$

where $q = \alpha^2/(\alpha + \beta)^2$. Then, the Lagrangian density can be written in terms of the potentials

$$\tilde{\phi} = \beta\phi + \alpha\hat{\phi}, \quad \bar{\phi} = \alpha\zeta(\phi - \hat{\phi}), \quad (12)$$

where $\zeta \equiv (\alpha + \beta)^{-1}$, as

$$\mathcal{L} = -\frac{1}{8\pi G} \{ \zeta (\vec{\nabla} \tilde{\phi})^2 + a_0^2 \tilde{\mathcal{M}}[(\vec{\nabla} \bar{\phi})^2/a_0^2] \} - \zeta \tilde{\phi} (\rho + \alpha\beta^{-1}\hat{\rho}) - \bar{\phi} (\rho - \hat{\rho}) + \frac{1}{2} (\rho \mathbf{v}^2 + \frac{\alpha}{\beta} \hat{\rho} \hat{\mathbf{v}}^2). \quad (13)$$

The two potentials are now decoupled, satisfying the equations

$$\Delta \tilde{\phi} = 4\pi G (\rho + \frac{\alpha}{\beta} \hat{\rho}), \quad \vec{\nabla} \cdot \{ \tilde{\mathcal{M}}'[(\vec{\nabla} \bar{\phi}/a_0)^2] \vec{\nabla} \bar{\phi} \} = 4\pi G (\rho - \hat{\rho}). \quad (14)$$

The potential $\tilde{\phi}$ is thus a linear combination

$$\tilde{\phi} = \phi^N + \frac{\alpha}{\beta} \hat{\phi}^N, \quad (15)$$

³In the present context, we can absorb $|\alpha/\beta|$ in $\hat{\rho}$; so, this choice of $|\gamma|$ may be viewed as the choice of a convenient normalization for $\hat{\rho}$.

where the Newtonian potentials in the two sectors, ϕ^N and $\hat{\phi}^N$, are the solutions of the Poisson equation for ρ and $\hat{\rho}$ separately. The matter and TM MOND potentials are then gotten as linear combinations

$$\phi = \zeta\tilde{\phi} + \bar{\phi} = (1+\lambda)^{-1}(\alpha^{-1}\phi^N + \beta^{-1}\hat{\phi}^N) + \bar{\phi}, \quad \hat{\phi} = \zeta\tilde{\phi} - \frac{\beta}{\alpha}\bar{\phi} = (1+\lambda)^{-1}(\alpha^{-1}\phi^N + \beta^{-1}\hat{\phi}^N) - \lambda\bar{\phi}, \quad (16)$$

where $\lambda = \beta/\alpha$.

The deep-MOND limit is formally implemented by taking $a_0 \rightarrow \infty$, $G \rightarrow 0$, with Ga_0 kept fixed. In this limit, $\bar{\phi}$ becomes super-dominant over the Newtonian-like potential $\tilde{\phi}$. We then have to have $\tilde{\mathcal{M}}'(w) \rightarrow w^{1/2}$ giving for the matter potential

$$\vec{\nabla} \cdot (|\vec{\nabla}\phi|\vec{\nabla}\phi) = 4\pi Ga_0(\rho - \hat{\rho}), \quad (17)$$

and $\hat{\phi} = -(\beta/\alpha)\phi$ for the TM potential, so we can also write

$$\vec{\nabla} \cdot (|\vec{\nabla}\hat{\phi}|\vec{\nabla}\hat{\phi}) = 4\pi Ga_0 \left(\frac{\beta}{\alpha}\right)^2 (\hat{\rho} - \rho). \quad (18)$$

If we put matter and TM test particles at the same position, \mathbf{r} , in a gravitational field, the former will have an acceleration $\mathbf{a} = -\vec{\nabla}\phi(\mathbf{r})$, while the latter will be accelerated by $-(\beta/\alpha)\mathbf{a}$. (Matter and TM do not follow the same weak equivalence principle, as is clear from the very construction of BIMOND, with matter following geodesics of $g_{\mu\nu}$ and TM those of $\hat{g}_{\mu\nu}$.) In the deep-MOND limit of theories with α and β of the same sign, matter attracts matter as in standard MOND; TM attracts TM as in MOND, but with an effective value of $Ga_0 \rightarrow Ga_0(\beta/\alpha)^2$; however, matter and TM repel each other.

To get the Newtonian behavior in the matter sector for $a_0 \rightarrow 0$; namely, to get $\phi \rightarrow \phi^N$ in this limit, for a pure-matter system,⁴ we have to have⁵ $\tilde{\mathcal{M}}'(\infty) = 1/(1 - \zeta)$.

Since $\tilde{\mathcal{M}}'(w) \approx w^{1/2}$ for small $w > 0$, and is thus necessarily positive there, we cannot have $\zeta > 1$, lest $\tilde{\mathcal{M}}'(w)$ vanishes at a finite value w , which it must not. This is the condition derived in Milgrom (2009c) on the same grounds. We can have $\zeta = 1$ (and then $\tilde{\mathcal{M}}' \rightarrow \infty$ at infinity); but this is excluded by solar-system constraints. There may be other constraints on the values of α , β from different consistency requirements in the relativistic and NR theories, but such are yet to be found.

With the above value of $\tilde{\mathcal{M}}'(\infty)$, we have in the high-acceleration regime

$$\phi = \phi^N + (\beta^{-1} - 1)\hat{\phi}^N, \quad \hat{\phi} = \left[1 + \frac{(\beta - 1)(\beta - \alpha)}{\alpha\beta}\right]\hat{\phi}^N + \frac{1 - \beta}{\alpha}\phi^N. \quad (19)$$

⁴Note that the Newtonian limit is defined by the requirement that it reproduces Newtonian dynamics in the Matter sector, not necessarily in the TM sector.

⁵This has to be required only because we insisted that G is Newton's constant. Had we started from some general coupling, G' , this relation would just constitute a definition of G in terms of G' .

We see that while G is the gravitational constant in the Newtonian limit of the matter sector, it is $\hat{G} = G[1 + (\beta - 1)(\beta - \alpha)/\alpha\beta]$ that plays this role in the TM sector. For $\beta = 1$ or $\beta = \alpha$ we have $\hat{G} = G$ but otherwise they are different. In fact, for some choices of α, β we have $\hat{G} < 0$, in which case the $a_0 \rightarrow 0$ limit corresponds to repulsive gravity in the TM sector. If we deem this undesirable we can eliminate the corresponding α, β values. (For $\alpha, \beta > 0$ we have $\hat{G} > 0$ with our already assumed inequality $\zeta < 1$.)

We also see from eq.(19) that for $\beta = 1$, $\phi = \phi^N$ and $\hat{\phi} = \hat{\phi}^N$, which means that there is no interaction between the two sectors in the Newtonian regime. This is true for the fully relativistic theory, where for $\beta = 1$ (irrespective of α) BIMOND separates in the limit $a_0 \rightarrow 0$ to two copies of GR in the two sectors [possibly with a cosmological constant $\sim a_0^2 \mathcal{M}(\infty)$].

When $\alpha \neq \beta$ the dynamics in the two sectors can be quite different. Such theories are worth investigating, but my purpose here is not to conduct an exhaustive study of this class of theories, only to demonstrate some salient results. To this end I shall consider mainly theories that are fully symmetric in matter-TM, namely those with $\alpha = \beta$. I will then discuss briefly some asymmetric cases.

2.1. A quasi-linear formulation

Solution of the field equations (14) requires solving a nonlinear Poisson equation for the given matter-TM configuration at hand. This may be rather taxing, especially when applying the theory to time-dependent problems, such as that of large-scale-structure formation. As in the case of the quasi-linear formulation of MOND (QUMOND; Milgrom 2009c), which parallels the Modified-Poisson-equation formulation of Bekenstein & Milgrom (1984), I describe here a quasi-linear theory, derivable from an action, not equivalent to our theory, but which captures much of its essence, and which should be much easier to apply. As in the case of QUMOND, this requires adding an auxiliary potential ψ to the MOND potentials for matter and TM.⁶ Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{8\pi G} \{2\alpha\zeta(\vec{\nabla}\phi - \vec{\nabla}\hat{\phi}) \cdot \vec{\nabla}\psi + \zeta(\beta\vec{\nabla}\phi + \alpha\vec{\nabla}\hat{\phi})^2 - a_0^2 \mathcal{Q}[(\vec{\nabla}\psi/a_0)^2]\} + \rho(\frac{1}{2}\mathbf{v}^2 - \phi) + \frac{\alpha}{\beta}\hat{\rho}(\frac{1}{2}\hat{\mathbf{v}}^2 - \hat{\phi}). \quad (20)$$

In terms of the two potentials $\bar{\phi}$ and $\tilde{\phi}$ related to the MOND potentials by eq.(12) one can write

$$\mathcal{L} = -\frac{1}{8\pi G} \{2\vec{\nabla}\bar{\phi} \cdot \vec{\nabla}\psi + \zeta(\vec{\nabla}\tilde{\phi})^2 - a_0^2 \mathcal{Q}[(\vec{\nabla}\psi/a_0)^2]\} - \bar{\phi}(\rho - \hat{\rho}) - \tilde{\phi}\zeta(\rho + \frac{\alpha}{\beta}\hat{\rho}) + \frac{1}{2}\rho\mathbf{v}^2 + \frac{\alpha}{2\beta}\hat{\rho}\hat{\mathbf{v}}^2, \quad (21)$$

giving the field equations

$$\Delta\tilde{\phi} = 4\pi G(\rho + \frac{\alpha}{\beta}\hat{\rho}), \quad \Delta\psi = 4\pi G(\rho - \hat{\rho}), \quad \Delta\bar{\phi} = \vec{\nabla} \cdot \{Q'[(\vec{\nabla}\psi/a_0)^2]\vec{\nabla}\psi\}. \quad (22)$$

⁶Since $\bar{\phi}$ and $\tilde{\phi}$ are decoupled in the Lagrangian density in eq.(13), we simply apply to $\bar{\phi}$ the same procedure that has lead to QUMOND.

It is easy to see that with the appropriate choice of \mathcal{Q} [namely, with $\tilde{\mathcal{M}}'(x^2)x = y$ being equivalent to $\mathcal{Q}'(y^2)y = x$] the theory is a very good mimic of equations (14). For example, in a spherically symmetric case they are identical.

3. Dynamics in the fully symmetric theory

In the fully matter-TM symmetric case, $\alpha = \beta$, which seems to give appealing cosmological solutions, we can write the field equations (14) as

$$\Delta\tilde{\phi} = 4\pi G(\rho + \hat{\rho}), \quad \vec{\nabla} \cdot \{\tilde{\mathcal{M}}'[(\vec{\nabla}\bar{\phi}/a_0)^2]\vec{\nabla}\bar{\phi}\} = 4\pi G(\rho - \hat{\rho}), \quad (23)$$

and the MOND potentials are then given by

$$\phi = \zeta\tilde{\phi} + \bar{\phi}, \quad \hat{\phi} = \zeta\tilde{\phi} - \bar{\phi}, \quad (24)$$

where now $\zeta = (2\beta)^{-1}$.

In the deep-MOND limit ϕ satisfies eq.(17), and $\hat{\phi} = -\phi$.

Interestingly, in the deep-MOND limit of the theory, TM behaves as if it has a negative active and passive gravitational mass relative to matter, while its inertial mass is still positive: Negative active mass because $\hat{\rho}$ enters the source for the matter ϕ potential with a negative sign, and negative passive mass, because it is accelerated by the gravitational field $\hat{\phi} = -\phi$. So bodies in the same sector attract each other, while bodies in different sectors repel each other. I discuss gravity in this important MOND limit in more detail in subsection 3.2.

The case $\rho \approx \hat{\rho}$ has to be commented on. In this case $\bar{\phi} \approx 0$, while $\tilde{\phi} \approx 2\phi^N \approx 2\hat{\phi}^N$. The MOND limit applies when $[Ga_0(\rho - \hat{\rho})R]^{1/2} \gg G(\rho + \hat{\rho})R$, where R is the characteristic size of the system. The configuration of density near-equality is unstable, and with a small separation, repulsion occurs (see subsection 3.3).⁷

Consider now the Newtonian limit, $a_0 \rightarrow 0$, where from the general eq.(19) we have

$$\phi = \phi^N + (2\zeta - 1)\hat{\phi}^N, \quad \hat{\phi} = \hat{\phi}^N + (2\zeta - 1)\phi^N. \quad (25)$$

We see that, unlike the MOND limit, here the predicted fields do depend on ζ . Recall that for the choice, $\zeta = 1/2$ ($\beta = 1$), we have $\phi = \phi^N$, $\hat{\phi} = \hat{\phi}^N$, with each sector seeing exactly its own

⁷A similar result applies in the more general, relativistic case: In configurations where the energy-momentum tensors in the two sectors are equal (with our normalization $\gamma = \alpha/\beta$) the solution of the field equations is $\hat{g}_{\mu\nu} = g_{\mu\nu}$, with both being the solution of the standard Einstein equation for the configuration [with a cosmological constant $\propto a_0^2\mathcal{M}(0)$]. For example, a double Schwarzschild metric is a spherically symmetric, vacuum solution of the BIMOND equations. Matching it to an interior solution will show that it corresponds to a central mass made of equal amounts of matter and TM.

Newtonian potential; so, matter and TM do not interact at all. For $\zeta > 1/2$ matter sees its own Newtonian potential plus a fraction $2\zeta - 1 > 0$ of that of TM; so they attract each other with a reduced effective gravitational constant. For $\zeta < 1/2$ they repel each other.

The case $\zeta \rightarrow 0$ ($\beta \rightarrow \infty$) is also interesting: We see from eq.(24) that in this case $\phi = -\hat{\phi} = \bar{\phi}$ for the full range of the theory (Newtonian-MOND); $\bar{\phi}$ becomes immaterial, and the MOND potential ϕ satisfies the second of eq.(23). We saw that for $\zeta = 0$ we have $\tilde{\mathcal{M}}'(\infty) = 1$. So in this case we end up with the MOND theory of Bekenstein & Milgrom (1984) with $\mu(x) = \tilde{\mathcal{M}}'(x^2)$, but with TM entering the theory as having a negative gravitational mass.

3.1. Forces on bodies

The force \mathbf{F} ($\hat{\mathbf{F}}$) on a matter (TM) body that constitutes a subsystem of the density ρ ($\hat{\rho}$) within the volume v (\hat{v}) is

$$\mathbf{F} = - \int_v \rho \vec{\nabla} \phi d^3r, \quad \hat{\mathbf{F}} = - \int_{\hat{v}} \hat{\rho} \vec{\nabla} \hat{\phi} d^3r. \quad (26)$$

Because the theory is nonlinear we cannot use in these expressions the potential produced by the system with the body in question excluded (as can be done in the linear case). As a result, even for a point mass m at position \mathbf{r} we cannot write the force simply as $\mathbf{F} = -m\vec{\nabla}\phi(\mathbf{r})$, where ϕ is produced by the rest of the system. In fact, the force is not even linear in the mass of the body, and becomes so only for test particles. In Milgrom (1997, 2002a), I discussed in detail general properties of forces in such nonlinear theories. Here I essentially use the results from these papers.

On dimensional grounds we can write the force between two matter point masses, M and m , a distance r apart, as $F = -a_0 M f(m/M, r/R_M)$, where $R_M = (MG/a_0)^{1/2}$ is the MOND radius for the mass M , and a negative sign signifies attraction. The same expression applies to the force between two TM point masses. For a matter point mass M and a TM point mass \hat{M} we can write $F = a_0 M f^*(\hat{M}/M, r/R_M)$.

In the test-particle limit, where one of the masses is much smaller than the other, $f(q, \lambda)$ and $f^*(q, \lambda)$ are easy to obtain for all values of λ , because the force on a test particle equals its mass times the gradient of the potential produced by the massive point mass, which can be gotten analytically. In the Newtonian limit $\lambda \ll 1$ we see from eq.(25) that $f(q, \lambda \ll 1) \approx q\lambda^{-2}$, which reproduces the Newtonian force, and $f^*(q, \lambda \ll 1) \approx (2\zeta - 1)q\lambda^{-2}$. Expressions for these functions in the deep-MOND limit $\lambda \gg 1$ will be given in subsection 3.2

3.2. Gravitational interactions in the deep-MOND regime

I now discuss in more detail matter-TM gravity in the deep-MOND regime of the symmetric theories. Look at a system of matter (ρ) and TM ($\hat{\rho}$) where the surface densities are everywhere

small enough that the accelerations everywhere are much smaller than a_0 .⁸ In this case, the potential field is determined from eq.(17), and the potential felt by matter is ϕ itself while that felt by TM is $-\phi$. To the MOND forces determined from these potentials one adds the subordinate Newtonian forces, coming from the $\tilde{\phi}$ potential, weighted by ζ , as per eq.(24).

Consider now forces on non-test bodies of both types. In Milgrom (1997), I showed that the theory described by eq.(17) is invariant under conformal transformations, just as the two-dimensional Poisson equation is. This enables us to derive some very useful exact results, which are otherwise difficult to obtain in nonlinear theories of the type discussed here. I also note, in passing, that the full symmetry of the theory is then the same as the isometry of a de Sitter space-time, with possible ramifications discussed in Milgrom (2009d). In addition to the usual invariance to translations and rotations, which have the usual consequences (e.g. conservation of momentum and angular momentum), the conformal symmetry in 3-dimensional Euclidean space includes inversions about a sphere of any radius R , centered at any point \mathbf{r}_0 . Namely, transformations of the form $\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r}_0 + R^2|\mathbf{r} - \mathbf{r}_0|^{-2}(\mathbf{r} - \mathbf{r}_0)$; and also dilatations $\mathbf{r} \rightarrow \mathbf{r}' = \lambda\mathbf{r}$. The symmetry means that if $\phi(\mathbf{r})$ is the MOND potential for mass distributions $\rho(\mathbf{r})$, $\hat{\rho}(\mathbf{r})$, then $\phi^t(\mathbf{r}') = \phi(\mathbf{r})$ is the MOND potential for mass distributions $\rho^t(\mathbf{r}') = J^{-1}\rho(\mathbf{r})$, $\hat{\rho}^t(\mathbf{r}') = J^{-1}\hat{\rho}(\mathbf{r})$, where $J(\mathbf{r}')$ is the Jacobian of the transformation $J = \|\partial\mathbf{r}'/\partial\mathbf{r}\|$. For example, from dilatation invariance follows that if $\phi(\mathbf{r})$ is the MOND potential for $\rho(\mathbf{r})$, $\hat{\rho}(\mathbf{r})$, then $\phi(\mathbf{r}/\lambda)$ is the MOND potential for mass distributions $\lambda^{-3}\rho(\mathbf{r}/\lambda)$, $\lambda^{-3}\hat{\rho}(\mathbf{r}/\lambda)$.

The existence of gravitational masses with opposite signs fits very naturally in such a conformally invariant theory. To take advantage of the symmetry, the point at infinity has to be treated on a par with all other points, since inversions, which are conformal transformations, interchange the point at infinity with a finite point. This implies that we should view the Euclidean space as the 3-dimensional sphere from the topological point of view. On the other hand, the gravitational field lines going to infinity away from a system of finite total mass, M , converge at infinity with a net flux, implying an effective negative mass $-M$, there. Under inversions this mass is brought to a finite point, leaving us with a configuration involving a negative mass, even if we start without one. So, having such masses from the start is an advantage. Put differently: a compact manifold such as the Euclidean sphere, as our space has to be viewed in a conformally invariant theory (here, to space transformation, not space-time), cannot accommodate a finite total mass: Applying Gauss theorem to any surface that separates space in two, implies that the total mass on one side has to be equal and opposite that on the other side, hence negative masses are needed.

Next, I describe several corollaries of the conformal symmetry. They pertain to a system made of matter and TM point masses, m_i , \hat{m}_k (all defined as positive), at positions \mathbf{r}_i , $\hat{\mathbf{r}}_k$, the forces on

⁸For our results to apply it is enough that the accelerations are small across the typical inter-particle distance in the system. For example the bodies we treat may be point masses, so the acceleration field near them is higher than a_0 . All we require then is that the Newtonian regimes of the different masses stay far away from each other. The point masses can then be taken as spheres larger than their MOND radii.

which are \mathbf{F}_i , $\hat{\mathbf{F}}_k$. Translation invariance dictates $\sum_i \mathbf{F}_i + \sum_k \hat{\mathbf{F}}_k = 0$, and rotational invariance implies $\sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_k \hat{\mathbf{r}}_k \times \hat{\mathbf{F}}_k = 0$.

(i) A ‘virial relation’ holds, which takes the form

$$\sum_i \mathbf{r}_i \cdot \mathbf{F}_i + \sum_k \hat{\mathbf{r}}_k \cdot \hat{\mathbf{F}}_k = -\frac{2}{3}(a_0 G)^{1/2} \left[\sum_i m_i - \sum_k \hat{m}_k \right]^{3/2} - \sum_i m_i^{3/2} - \sum_k \hat{m}_k^{3/2}. \quad (27)$$

There is an additional relation that applies for systems with vanishing total ‘charge’ (see Milgrom 1997).

(ii) The following are corollaries of relation (27) for the two body case: For two matter masses, say M at the origin and m at \mathbf{r} , the force on m is

$$\mathbf{F} = -\frac{2}{3}(a_0 G)^{1/2} [(M + m)^{3/2} - M^{3/2} - m^{3/2}] \frac{\mathbf{r}}{r^2}; \quad (28)$$

so $f(q, \lambda \gg 1) \approx (2/3)\lambda^{-1}[(1 + q)^{3/2} - 1 - q^{3/2}]$. This force, which is the same for two TM masses, is always attractive. For a matter mass M at the origin and a TM mass \hat{M} at \mathbf{r} , the (repulsive) force on \hat{M} is

$$\hat{\mathbf{F}} = \frac{2}{3}(a_0 G)^{1/2} [M^{3/2} + \hat{M}^{3/2} - |M - \hat{M}|^{3/2}] \frac{\mathbf{r}}{r^2}; \quad (29)$$

so, $f^*(q, \lambda \gg 1) \approx (2/3)\lambda^{-1}[1 + q^{3/2} - (1 - q)^{3/2}]$. For $\hat{M} = M$, $\hat{\mathbf{F}} = (4/3)(a_0 G)^{1/2} M^{3/2}(\mathbf{r}/r^2)$. This is stronger by a factor $1/(2^{1/2} - 1) \approx 2.4$ than the attracting force between two equal matter or TM masses.

(iii) The theory is nonlinear; so forces are not additive: When we have more bodies present, the force on each has to be determined through a new calculation of the whole system. This, to my knowledge, is impossible to derive analytically, in general, but there are exceptions, several examples are discussed below to demonstrate that matter-TM repulsion is general (see also Milgrom 2002a for general theorems on this issue).

(iv) The forces in a three-body system with ‘vanishing total charge’; i.e., with $\sum_i m_i = \sum_k \hat{m}_k$, was calculated in Milgrom (1997). The force on one of the bodies, call it 1, is

$$\mathbf{f}_1 = \beta_{12} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^2} + \beta_{13} \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^2}, \quad (30)$$

where $\beta_{ij} = (2/3)(a_0 G)^{1/2}(|q_i + q_j|^{3/2} - |q_i|^{3/2} - |q_j|^{3/2})$, with $q_i = m_i$ for a matter body, and $q_k = -\hat{m}_k$ for a TM body. So, interestingly, in this case the force is simply the sum of the two forces that would have applied by the two bodies separately.

(v) Other configurations for which there are analytic results involve symmetric configuration. For example, take a configuration of N equal masses M , of the same type, symmetrically placed at a distance r from the origin, so that the forces on all are equal and radial (such as at the corners of a square, a cube, etc., or forming a uniform spherical shell in the limit of large N). Application of relation (27) give the force on each mass as $-(2/3)M(MGa_0)^{1/2}(N^{3/2} - N)\mathbf{r}r^{-2}$. Now place any

mass m of the opposite type at the center and the force becomes $-(2/3)(Ga_0)^{1/2}(|NM - m|^{3/2} - NM^{3/2} - m^{3/2})\mathbf{r}r^{-2}$. The addition of m clearly weakens the self attraction towards the origin, and reverses the sign of the force for large enough m .

In a similar fashion, if we have an equal number of matter and TM point bodies of the same mass arranged symmetrically, so that all forces act radially and are equal (for example two masses of each type on alternate corners of a square), relation (27) tells us that they are all repelled from the center.

(vi) The potential field of a pair of equal matter and TM masses, M , at \mathbf{r}_1 and \mathbf{r}_2 , respectively (in the MOND regime) is

$$\phi(\mathbf{r}) = (MGa_0)^{1/2} \ln \frac{|\mathbf{r} - \mathbf{r}_1|}{|\mathbf{r} - \mathbf{r}_2|}. \quad (31)$$

(vii) More generally, all the physics of a system of gravitating point ‘charges’, q_i , is encapsulated in the N -mass energy functions $E(\mathbf{r}_1, \dots, \mathbf{r}_N)$, since the forces are derived from them through $\mathbf{F}_i = -\partial E / \partial \mathbf{r}_i$ (The force on a given mass measures the change in the energy under a rigid translation of the mass.)⁹ The above results give us the analytic expressions for E for the general two-body system:

$$E(\mathbf{r}_1, \mathbf{r}_2) = \ln |\mathbf{r}_1 - \mathbf{r}_2|^{\beta_{12}}, \quad (32)$$

and for a three-body system with vanishing total ‘charge’:

$$E(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \ln [r_{12}^{\beta_{12}} r_{13}^{\beta_{13}} r_{23}^{\beta_{23}}], \quad (33)$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ [Relation (32) is, in fact, a special case of this with a renormalized energy for the limit where mass 3 is sent to infinity ($\mathbf{r}_3 \rightarrow \infty$)]. We do not know the general form of E . For systems with vanishing total ‘charge’ we can show that $E(\mathbf{r}_1, \dots, \mathbf{r}_N)$ is determined up to an unknown function of the $N(N-3)/2$ conformally invariant variables $u_{ijkl} = r_{ij}r_{kl}/r_{ik}r_{jl}$:

$$E(\mathbf{r}_1, \dots, \mathbf{r}_N) = E_0(u_{ijkl}) + \sum_{1 \leq i < j \leq N} b_{ij}^N \ln r_{ij}, \quad (34)$$

where (for $N > 2$)

$$b_{ij}^N = \frac{2}{(N-2)} \left[-\nu(q_i) - \nu(q_j) + \frac{1}{(N-1)} \sum_{m=1}^N \nu(q_m) \right], \quad (35)$$

with $\nu(q) = (2/3)(Ga_0)^{1/2}|q|^{3/2}$. For $N = 3$ we have no u_{ijkl} variables, so E_0 is a constant and $\beta_{ij} = b_{ij}^3$. Both E and E_0 have a suppressed dependence on the charges.¹⁰ The second term in

⁹The gravitational energy of an isolated system with non-vanishing total ‘charge’ is infinite (it is finite for a vanishing total ‘charge’), but its difference for two systems with the same ‘charge’ are finite, and only these interest us.

¹⁰In D dimensions $\nu(q) \propto q^{D/(D-1)}$. In two dimensions we have the exact result $E(\mathbf{r}_1, \dots, \mathbf{r}_N) \propto \sum_{i < j} q_i q_j \ln r_{ij}$.

eq.(34) carries the anomalous transformation properties of the energy under dilatations and special conformal transformations, while E_0 is truly invariant.¹¹ For example, under dilatations

$$E(\lambda \mathbf{r}_1, \dots, \lambda \mathbf{r}_N) = E(\mathbf{r}_1, \dots, \mathbf{r}_N) - \ln \lambda \sum_{m=1}^N \nu(q_m). \quad (36)$$

Again, by sending one of the charges to infinity (call it $q_N = -Q$), and renormalizing the energy by subtracting the constant $\ln(r_N) \sum_1^{N-1} b_{iN}^N = -2\nu(Q)\ln(r_N)$, which goes to infinity in the limit, we can define the energy for a system of charges with a finite total charge Q ; for example, a system with only matter masses (good only for comparing systems with this total charge). The sum in eq.(34) then goes only up to $N - 1$, and the u variables with index N are written, e.g., $u_{ijN} = r_{ij}/r_{iN}$; so they are still invariant under dilatations (but not inversions); so E_0 is invariant under dilatations of $\mathbf{r}_1, \dots, \mathbf{r}_{N-1}$. The energy for the remaining $N - 1$ masses transforms under dilatations as:

$$E(\lambda \mathbf{r}_1, \dots, \lambda \mathbf{r}_{N-1}) = E(\mathbf{r}_1, \dots, \mathbf{r}_{N-1}) + A \ln \lambda, \quad (37)$$

where

$$A = \sum_{1 \leq i < j \leq N-1} b_{ij}^N = \nu(Q) - \sum_{m=1}^{N-1} \nu(q_m); \quad (38)$$

this is equivalent to eq.(27) (seen by taking the λ derivative at $\lambda = 1$).

(viii) In general, the forces in two systems that are related by a conformal transformation of the charges are simply related (see Milgrom 1997). For example, consider a point charge q at a distance $r < a$ from the center of a spherical shell of radius a , uniformly charged by $-q$. On dimensional grounds, the force on the point mass can be written as $f(r) = -2\nu(q)ra^{-2}s(r/a)$. Then, from conformal invariance, the force on q when it is at $R > a$ is $f(R) = 2\nu(q)R^{-1}[1 + (a/R)^2s(a/R)]$ (the function s is not known).

Under conformal transformations, equipotential surfaces go to equipotential surfaces of the new configuration, and field lines go to field lines. Also, spheres (circles) go to spheres (circles), including planes (straight lines), which are spheres of infinite radius. For an arbitrary system of charges, q_i , with $\sum q_i = 0$, lying on a sphere of radius a , one can show¹² that the radial component of the force on q_i is $\nu(q_i)/a$ (pointing outwards), and everywhere on the sphere, outside the charges, the field lines are tangent to the sphere (no radial force on test charges).

¹¹The energy of a zero-total-charge system is invariant to all conformal transformations of the charge distribution. However, here we speak of transforming the point charges rigidly to their new positions, without affecting the transformation on their internal structure. The anomalous transformation properties of $E(\mathbf{r}_1, \dots, \mathbf{r}_N)$ result from this and the fact that the energy of a finite-charge system does change under dilatations, $\mathbf{r} \rightarrow \lambda \mathbf{r}$, by $\nu(q)\ln\lambda$.

¹²This is done by transforming the sphere into a plane where the forces must all lie from symmetry, and using the transformation law for forces from Milgrom (1997).

3.3. Well mixed configurations

In subsection 3.2, I considered mainly systems of point masses for which matter and TM are well separated. It is worth noting that interesting situations may occur when matter and TM are well mixed in the sense that $\hat{\rho} \approx \rho$.

Look, for example, at the case $\hat{\rho} = \rho$: In this case $\bar{\phi} = 0$, and $\tilde{\phi}$ is twice the Newtonian potential for each mass separately, call it ϕ^N . So the two sectors see the same potential, $\phi = \hat{\phi} = 2\zeta\phi^N$. Given also the same initial velocities, such mass configurations will retain their density equality with time, with each mass type developing according to Newtonian dynamics with an effective Newton constant G/β . This echoes the situation in symmetric cosmology, where we have $\hat{g}_{\mu\nu} = g_{\mu\nu}$, with each satisfying Friedmann's equations [possibly with a cosmological constant $\sim \mathcal{M}(0)a_0^2$] and with an effective gravitational constant G/β . Such local configurations are, however, unstable, and a small departure from equal densities will lead to eventual segregation of the two mass types. For example, in the presence of a small matter, or TM, mass ρ^* (or with a small departure from $\rho = \hat{\rho}$) a nonzero $\bar{\phi}$ potential is created which is added to ϕ and $\hat{\phi}$ with opposite signs, causing the separation of ρ from $\hat{\rho}$ in a way that increases the separating force even further. Thus initially, at least, even if the surface densities of ρ and $\hat{\rho}$ separately are large compared with a_0/G , so that each mass type creates a high-acceleration Newtonian field, the resulting evolution of the difference $\delta = \rho - \hat{\rho}$ will be governed by MOND, and is nonlinear.

3.4. Relativistic point mass solutions

Allowing for TM in BIMOND permits a two-parameter (M, \hat{M}) family of (relativistic) spherically symmetric, static, vacuum solutions on a double Minkowski background (neglecting ‘cosmological constant’ effects). These would describe, e.g., matter-TM black holes. For $M = \hat{M}$ we have the solutions $\hat{g}_{\mu\nu} = g_{\mu\nu}$, with both metrics being of the Schwarzschild form for M , with G/β as gravitational constant. At the other end, for $\hat{M} = 0$ we can have pure-matter black holes, for instance. Because of the cosmological coincidence $a_0 \approx cH_0/2\pi$, for all sub-Universe systems the horizon acceleration is much larger than a_0 . This means that MOND effects enter only far beyond the horizon, deep in the NR regime. For $\beta = 1$ we know that in the high-acceleration regime BIMOND describes two separate Einstein theories for matter and TM [with a cosmological constant $\sim a_0^2\mathcal{M}(\infty)$, which I neglect here]. In this case, each of the metrics is nearly a Schwarzschild one corresponding to its own mass, within many Schwarzschild radii. At large radii the metrics approach their NR expressions as per eqs.(4)(23-24): at first $\phi \approx -MG/r$, $\hat{\phi} \approx -\hat{M}G/r$, and only at much larger radii $\phi \approx -\hat{\phi} \approx \bar{\phi} \approx [(M - \hat{M})Ga_0]^{1/2} \ln r$ (for $M \geq \hat{M}$).

3.5. Asymptotic field of equal-mass systems

For an isolated system with equal total masses of matter and TM, $M = \hat{M}$, the asymptotic behavior of the MOND potential is not described by the usual logarithmic dependence on the radius. This can be easily seen by applying the Gauss theorem to eq.(17), which implies a vanishing coefficient to a logarithmic term. Instead, the generic asymptotic potential is of the form

$$\phi(\mathbf{r}) \rightarrow (MGa_0)^{1/2} \mathbf{R} \cdot \frac{\mathbf{r}}{r^2}, \quad (39)$$

where \mathbf{R} is some radius vector characteristic of the mass distribution in the system. This is the vacuum solution of eq.(17) with the slowest decrease with radius. It is gotten by a conformal transformation (inversion at the origin) from the configuration of a constant acceleration field $\phi \propto \mathbf{R} \cdot \mathbf{r}$, which is clearly a vacuum solution, and is thus itself a vacuum solution by virtue of the conformal invariance of eq.(17) (Milgrom 1997).¹³ Not surprisingly, the potential in eq.(39) has the form of a dipolar electrostatic field in two dimensions: $\mathbf{D} \cdot \mathbf{r}/r^2$. However, here the asymptotic “dipole strength” is not related to the dipole of the mass distribution; in fact, I do not know how to express \mathbf{R} in terms of the mass distribution. For a matter-TM point-mass pair, the exact expression, eq.(31), implies $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$. More generally, we know that \mathbf{R} does not scale with the mass, and scales with the system size: when $\rho(\mathbf{r}) \rightarrow \lambda^{-3}\rho(\mathbf{r}/\lambda)$, we have $\mathbf{R} \rightarrow \lambda\mathbf{R}$.

The magnitude of the asymptotic field in eq.(39), $|\vec{\nabla}\phi| = (MGa_0)^{1/2}|\mathbf{R}|r^{-2}$, depends only on r (not on the direction) and decreases with radius like the Newtonian field of the system $|\vec{\nabla}\tilde{\phi}| = 2MGr^{-2}$. Their ratio is $|\vec{\nabla}\tilde{\phi}|/|\vec{\nabla}\phi| = 2R_M/|\mathbf{R}|$, where R_M is the MOND radius of the system. So, if the mass distribution is such that $|\mathbf{R}| \gg R_M$, the MOND contribution dominates the field asymptotically as well.

There are configurations for which $|\mathbf{R}|$ is much smaller than the characteristic size of the system. In particular, there are systems for which $\mathbf{R} = 0$, for example due to symmetry. For these, the MOND potential decreases faster asymptotically, and the Newtonian field dominates there. I have not identified the next slowest decreasing vacuum solution of eq.(17). Because of the conformal invariance of the problem, finding such solutions is equivalent to finding the behavior of vacuum solutions near the symmetry point of a symmetric mass configuration. Take, for instance, a system with two matter masses, M , at opposite corners of a square, centered at the origin, and two TM masses $\hat{M} = M$ at the other two corners. Inversion about a sphere centered at the origin, and containing the corners, leaves the mass configuration invariant, but interchanges the origin with infinity. This implies that the behaviors of the potential near the origin and at infinity are gotten from each other by a simple inversion. Spherically symmetric configurations with $M = \hat{M}$ have a vanishing MOND potential ϕ outside the mass, with only $\tilde{\phi}$ contributing.

¹³The logarithmic vacuum solution transforms to itself.

4. Observational consequences

No BIMOND effects are felt in the cosmos, in the picture in which on cosmological scales $\hat{g}_{\mu\nu} = g_{\mu\nu}$, as long as matter is homogeneous (except perhaps those of a cosmological constant). BIMOND effects appear when inhomogeneities develop—the beginning of structure formation—when the presumably uncorrelated perturbations induced in the two sectors would have caused local departures from equality of the metrics. The initial stages of the growth of perturbations on the background of the expanding Universe would have been affected already by some of the peculiar matter-TM gravitational interaction discussed here, but these require using the relativistic version of the theory. Subsequent, nonlinear development of inhomogeneities occur already in the NR regime, for which we can directly use the NR theories, and the results of this paper. Matter-TM interactions can also lead to observable effects in the present day Universe. Here I discuss very succinctly some possible consequences of the existence of TM and its BIMOND interaction with matter.

4.0.1. Structure formation

Even if the underlying theory itself is symmetric in the two sectors, it is not clear that the attributes of their cosmic matter contents are exactly the same. For example, we do not yet know what engendered the baryon asymmetry in matter. It is possible, then, that the small initial asymmetry was different in the two sectors, which could result in different baryon densities today. Also, even if there occurred similar inflation episodes in the two sectors, they could have emerged from them with different perturbation amplitudes. The effects of such possible asymmetries, and others, on structure formation should be studied.

The presence of TM and its peculiar interactions with matter, discussed here, would have had major impact on the process of structure formation. As already mentioned in the introduction, in a symmetric universe, with $\hat{g}_{\mu\nu} = g_{\mu\nu}$ on large scales, small fluctuations are governed by nonlinear MOND dynamics, even at early times when the universe at large is characterized by high accelerations, $cH \gg a_0$. What dictates that we are in the MOND regime is the smallness of the perturbations, leading to small departures from metric equality between the two sectors: The argument of the interaction function \mathcal{M} , or its derivative \mathcal{M}' appearing in the field equations, is the difference in the connections of the two metrics in the relativistic regime, or the difference in the gradients of the matter and TM potentials in the NR regime, in units of a_0 . The growth of perturbations in this picture is thus nonlinear from the start, since the MOND potential is nonlinear in the overdensity. To apply the present results to the problem would thus require numerical simulations.

Simulations of structure formation with NR MOND dynamics has been considered in several studies; e.g., by Sanders (2001), Nusser (2002), Stachniewicz & Kutschera (2002), Knebe (2005), and Llinares, Knebe, & Zhao (2008). It should be easy to extend such simulation to reckon with

TM, by including an initial TM distribution with its own seed fluctuations, and employing the field equations (14)(16), or alternatively the more wieldy eqs.(22)(16).

Our results here are not directly applicable to the early stages of the growth of perturbations, because they assume that the system at hand is much smaller than the characteristic cosmological curvature radius, and they ignore the underlying cosmic expansion, relying on perturbations around Minkowski metrics, not those describing cosmology. However, these results would still apply at later times, and might be indicative of what happens qualitatively even at early times.

One important effect that can be anticipated is the segregation of matter and TM due to their mutual repulsion. This would presumably lead to formation of interweaving cosmic webs for matter and TM; i.e., complementary networks of nodes, filaments, and voids, with the high-density nodes of one sector residing in the voids of the other, and vice-versa, with filaments avoiding each other.

Some of the main questions that can be answered by numerical simulations are: Do matter and TM indeed segregate efficiently and what does the present-day configuration look like? Do they indeed form interweaving, mutually avoiding cosmic webs? Is segregation practically complete, or do we still find galaxy size or larger TM bodies lingering in the neighborhood of matter structures? Does this occur often enough for direct gravitational effects of TM on matter to be a common phenomenon? To what extent voids are more pronounced (empty) compared with simulations with matter only? What are the effects on peculiar motions of large-scale structures?

Regarding the question of void structure, in particular, it has been suggested that various observed aspects of voids pose problems for the LCDM paradigm (e.g., Peebles 2001, Tully 2007, Tully & al. 2008, Tikhonov & Klipin 2009, Tikhonov & al. 2009, and see van de Weygaert & Platen 2009, and Peebles & Nusser 2010 for recent reviews). Even without TM, MOND is expected to produce more pronounced voids than DM for the following reason: A void acts like a region of negative mass. If the ambient density is ρ_a , then a void can have at most the effective density $-\rho_a$. In MOND this affective density can be much higher if the characteristic accelerations involved are small enough. TM can help even more because of the added repulsion by TM concentrations, which would arguably reside in matter voids.

Simulations should also study the effects of various possible departures from symmetry between the two sectors, such as disparate baryon densities, different amplitudes of the initial fluctuations, etc. One should also investigate the dependence of the outcome on the parameter ζ , and also consider structure formation in asymmetric theories with $\alpha \neq \beta$.

4.0.2. *Gravitational lensing*

The best prospects for detecting or constraining TM, if it has formed structures like those of matter, and if it has segregated efficiently from matter, seem to be via weak gravitational lensing.

Expressions (4) for the two NR metrics imply that we can calculate lensing in the standard

way, as is done in GR, using the MOND potentials instead of the Newtonian potentials.¹⁴

We are interested here in the lensing effects of an isolated TM body, $\hat{\rho}$, on matter photons. These are dictated by the matter MOND potential ϕ created by $\hat{\rho}$. Use, as an example, the field equations in the form of eq.(8), with $\alpha = \beta = 1$, and $\rho = 0$. We then have

$$\Delta\phi = -4\pi G\hat{\rho}_p, \quad \Delta\hat{\phi} = 4\pi G(\hat{\rho} + \hat{\rho}_p), \quad (40)$$

where $\hat{\rho}_p = -\vec{\nabla} \cdot (\mathcal{M}' \vec{\nabla} \phi^*)$. In other words, we can calculate ϕ as the Newtonian potential produced by minus the phantom density produced by $\hat{\rho}$ alone. This phantom density is the same as we would calculate for a matter distribution $\rho = \hat{\rho}$. Thus lensing of matter photons by a TM body are different in two major ways from lensing by a matter body of the same mass distribution: First, photons (or any matter test particle) do not ‘see’ the TM itself, only the fictitious, MOND, phantom matter that it produces. Second, photons sense this phantom matter with an opposite sign: a TM lens thus acts as a diverging lens.

The first fact above implies that photons are oblivious to the existence of the TM in regions of high acceleration: For a spherical body with radially decreasing acceleration, no force is felt in the Newtonian regime. For example, a point TM mass \hat{M} , is not felt by matter roughly within its MOND radius $R_M = (\hat{M}G/a_0)^{1/2}$.

Another fact to recall is that the thin-lens approximation does not apply in MOND (Mortlock and Turner 2001, Milgrom 2002b, Milgrom & Sanders 2008) and structure along the line of sight has to be reckoned with: What enters lensing is still the integrated surface (column) density of the phantom matter. This, however, does not depend only on the surface density distribution of the baryonic matter that produces it; it also depends crucially on the baryon distribution along the lines of sight. For simplicity’s sake, I consider here only a spherical lens. In this case, the ratio of the Einstein radius of a lens to its MOND radius is

$$\frac{r_E}{R_M} = \left(\frac{4a_0 d_{ls} d_l}{c^2 d_s} \right)^{1/2}, \quad (41)$$

with d_{ls} , d_l , d_s the lens-source, lens, and source angular-diameter distances, respectively. Considering the well known proximity $a_0 \approx cH_0/2\pi$, this ratio is

$$\frac{r_E}{R_M} \approx \left(\frac{2d_{ls}d_l}{\pi d_s D_H} \right)^{1/2}, \quad (42)$$

where $D_H = c/H_0$ is the Hubble distance. Thus, for lenses nearer than the Hubble distance, the Einstein radius is within the MOND radius. But no bending occurs within the MOND radius, since \hat{M} itself is not felt, and there is no PM there. Also, for a spherical lens, the PM is by definition

¹⁴This is not the case for BIMOND theories that hinge on scalar arguments of the bimetric interaction other than that given in eq.(3).

always at or outside its MOND radius and so definitely outside its Einstein radius (the surface density of PM is always subcritical for a spherical body—see below). We thus do not expect strong lensing effects of TM on matter photons.¹⁵

Since only the phantom matter affects lensing of matter photons by TM, we should recall some of its important properties. I mentioned already that it is to be found only roughly beyond the MOND radius. Another pertinent attribute is that its local surface density (for a spherical system) is bounded by some universal value: $\Sigma \lesssim \Sigma_0 \approx a_0/2\pi G$. This is related to the observation of Brada & Milgrom (1999) that the acceleration produced by the MOND PM can never much exceed a_0 . More directly, I showed in Milgrom (2009a) that the MOND prediction for the central surface density of phantom haloes in spheroidal systems is $\lesssim \Sigma_0$. This fact is also supported by observations of galaxies (Donato & al. 2009, Gentile & al. 2009). The exact value of the maximum as deduced from MOND depends somewhat on the MOND formulation and on the choice of extrapolating function. In comparison with the lensing critical surface density of the lens $\Sigma_c \equiv c^2 d_s / 4\pi G d_l$ we have, using again the numerical proximity of $a_0 \approx cH_0/2\pi$,

$$\frac{\Sigma_0}{\Sigma_c} = \frac{d_{ls} d_l}{\pi d_s D_H}. \quad (43)$$

So PM lensing, and as a result TM lensing of matter photons, is always sub-critical for spherical lenses, and very much so for low redshift lenses.

A possible, interesting signature of lensing by the PM of a TM body is the appearance of a ring, or a projected shell of PM (negative mass in our case) if the TM body is contained within its MOND radius, as discussed in Milgrom & Sanders (2008). With lensing by a matter body (such as a galaxy cluster) the weak ring feature is generally masked by the dominant surface density of the baryonic body. But in the case of lensing by TM, the TM baryons do not contribute to lensing, and thus do not mask the feature if it is produced.

We should try then to identify the signature of possible TM bodies, such as TM galaxy clusters, in weak lensing surveys. Several factors may turn out to make such identification difficult: In the first place, we can, of course, not rely on guidance, or substantiating evidence, from matter effects—as can be done in standard weak lensing—such as x-ray emission, Sunyaev-Zel’dovich effect, or a direct view of a galaxy concentration. Second, as we saw, no high surface density effects are expected, since only the rather low-surface-density fictitious PM acts as a lens. Third, the signal should evince the effects of a negative mass concentration, which may require specialized algorithms to discover. Fourth, matter structures along the line of sight will act to cancel the lensing effects of TM bodies, or at least distort and confuse them.

¹⁵With a TM lens that is elongated along the line of sight we can have strong lensing. For example, consider N equal point TM masses, \hat{M} , along the line of sight, separated from each other by more than their individual MOND radius R_M . The projected PM is then N times the individual contribution, so the Einstein radius of the system is $N^{1/2}$ times that of a single mass. For large enough N , this is larger than R_M , and we can then get strong lensing effects. Also, with $\beta \neq 1$ there will be attractive or repulsive strong lensing effects.

Results of large-scale-structure simulations with TM will pinpoint the expectations for weak lensing by TM, and help design search strategies.

4.0.3. Other effects

If matter-TM segregation is not complete, and galaxy-size TM bodies are still lurking in the neighborhood of matter galaxies, they may produce distortions that are visible as warps, lopsidedness, or ellipticity. In particular, in theories with $\beta < 1$, matter and TM attract each other in the high acceleration regime; so, there might be TM trapped in high acceleration regimes of matter territory. One may speculate in such a case that small amounts of TM trapped in the (high acceleration) cores of galaxy clusters might be responsible for the observed mass discrepancy there (see 5.3 below).

If structure formation simulations confirm that indeed matter and TM form interleaving mutually avoiding webs, and if there is matter-TM symmetry in the cosmos, we can estimate the occurrence of TM galaxies in matter territory, by seeing how often we find matter galaxies in voids, which are presumably TM territories.

Clearly, there should also be important effects of the existence of TM on the appearance of the CMB fluctuations.

5. Asymmetric theories

In theories with $\alpha \neq \beta$ we have different gravitational dynamics in the two sectors. In particular, in such theories the amount of TM in the universe, and its general properties (distribution, etc.) can be rather different from those of matter. Such theories are, clearly, worth investigating, even if they involve treating less amenable configurations. Here I discuss briefly the NR limit of several examples of such theories.

5.1. A theory with $\beta = 1$ and $\alpha \gg 1$

In this example we take the extreme limit $\alpha \rightarrow \infty$ while $\beta = 1$. We can then write the field equations for the MOND potentials, eqs.(14-16), as

$$\Delta(\zeta\tilde{\phi}) = 4\pi G\hat{\rho}, \quad \vec{\nabla} \cdot \{\tilde{\mathcal{M}}'[(\vec{\nabla}\bar{\phi}/a_0)^2]\vec{\nabla}\bar{\phi}\} = 4\pi G(\rho - \hat{\rho}), \quad (44)$$

$$\phi = \zeta\tilde{\phi} + \bar{\phi}, \quad \hat{\phi} = \zeta\tilde{\phi}. \quad (45)$$

Since now $\zeta \rightarrow 0$, we have $\tilde{\mathcal{M}}'(\infty) = 1$. We find then that $\zeta\tilde{\phi} = \hat{\phi}^N$ is the Newtonian potential of $\hat{\rho}$, while $\bar{\phi}$ is a solution of the nonlinear Poisson equation $\vec{\nabla} \cdot [\mu(|\vec{\nabla}\bar{\phi}|/a_0)\vec{\nabla}\bar{\phi}] = 4\pi G(\rho - \hat{\rho})$, with μ

the standard MOND interpolating function, and $\phi = \bar{\phi} + \hat{\phi}^N$, $\hat{\phi} = \hat{\phi}^N$. Dynamics in the twin sector is thus fully Newtonian and oblivious to the matter sector. In contradistinction, the gravitational potential in the matter sector is governed by the sum of the MOND potential produced via the nonlinear Poisson equation by $\rho - \hat{\rho}$, with the Newtonian potential of twin matter.

5.2. A theory with $\alpha + \beta = 0$

As another example, take the case $\alpha + \beta = 0$, which leads to the quasi-linear formulation of MOND (QUMOND) discussed at length in Milgrom (2009c). The field equations are then

$$\begin{aligned}\Delta\phi &= 4\pi G\beta^{-1}\rho + \beta^{-1}\vec{\nabla} \cdot (\mathcal{M}'\vec{\nabla}\phi^*) = 4\pi G(\rho + \rho_p), \\ \Delta\hat{\phi} &= 4\pi G\beta^{-1}\hat{\rho} + \beta^{-1}\vec{\nabla} \cdot (\mathcal{M}'\vec{\nabla}\phi^*) = 4\pi G(\hat{\rho} + \hat{\rho}_p),\end{aligned}\tag{46}$$

where, $\phi^* = \phi - \hat{\phi}$, and

$$\begin{aligned}\rho_p &\equiv (4\pi G)^{-1}\beta^{-1}\vec{\nabla} \cdot (\mathcal{M}'\vec{\nabla}\phi^*) + (\beta^{-1} - 1)\rho, \\ \hat{\rho}_p &\equiv (4\pi G)^{-1}\beta^{-1}\vec{\nabla} \cdot (\mathcal{M}'\vec{\nabla}\phi^*) + (\beta^{-1} - 1)\hat{\rho}.\end{aligned}\tag{47}$$

Taking the difference of these equations gives

$$\Delta\phi^* = 4\pi G\beta^{-1}(\rho - \hat{\rho}).\tag{48}$$

One first solve the Poisson eq.(48) for ϕ^* , and then another Poisson equation for ϕ or $\hat{\phi}$. Accelerations of test particles are given by $\mathbf{a} = -\vec{\nabla}\phi$, $\hat{\mathbf{a}} = -\vec{\nabla}\hat{\phi}$.

We see that, again, the dynamics within each sector separately are not the same. If we compare the NR potentials for two configurations $\hat{\rho} = \rho_R$, $\rho = 0$, and $\rho = \rho_R$, $\hat{\rho} = 0$, we see that ϕ^* has an opposite sign for the two configurations, and hence $\hat{\rho}_p \neq \rho_p$.

The conserved momentum is now $\mathbf{P} = \int(\rho\mathbf{v} - \hat{\rho}\hat{\mathbf{v}})$. If we still define the force on a matter subsystem in the volume v as $\mathbf{F} = -\int_v \rho\vec{\nabla}\phi d^3r$, and that on TM as $\hat{\mathbf{F}} = -\int_v \hat{\rho}\vec{\nabla}\hat{\phi} d^3r$, then for a closed system made of matter and TM it is $\mathbf{F} - \hat{\mathbf{F}}$ that vanishes, not their sum. If they do not vanish separately, such an isolated system will self accelerate (still preserving momentum). This seems paradoxical if applied to an isolated system, but in the context of a universe filled with matter and TM in equal amount it does not necessarily lead to unacceptable behavior since no large scale directed accelerations can occur. It remains to be checked by numerical simulations whether this can lead to inconsistencies (conceptual or observational) in the expected behavior of matter.

Specialize further to $\beta = 1$, which is particularly transparent: In this case eq.(48) implies that ϕ^* is the Newtonian potential of the system with TM contributing as having negative (active) gravitational mass. In eq.(47) we have $\rho_p = \hat{\rho}_p = (4\pi G)^{-1}\vec{\nabla} \cdot (\mathcal{M}'\vec{\nabla}\phi^*)$. To get a Newtonian limit in the matter sector we have to have $\mathcal{M}'(z) \rightarrow 0$ for $z \rightarrow \infty$ [$z = (\vec{\nabla}\phi^*/a_0)^2$]. This also gives standard

Newtonian dynamics in the TM sector. It also means that matter and TM do not interact in the deep Newtonian regime. In this case the total force on matter and on TM for a closed system vanish separately.

To get MOND dynamics in the matter sector we have to have $\mathcal{M}'(z) \approx z^{-1/4}$ for $z \ll 1$. Thus effectively, it can be said that we can use a Newtonian calculation with each type of mass ‘seeing’ the density produced by its own type plus the phantom mass that is common to both sectors. Because $\hat{\rho}$ enters the source of ϕ^* with an opposite sign one may say that the phantom mass produced by TM alone is repulsive to both types of matter, while that produced by matter is attractive to both. Thus, in the TM sector, a point mass \hat{M} produces a Newtonian, attractive force on a TM test particle within its MOND radius, R_M , but instead of the MOND enhancement of gravitational attraction in the matter sector, in the TM sector MOND effects cause this force to weaken the attraction, and turn it into repulsion at large distances—a manifestation of the disparate behavior in the two sectors.

5.3. A theory with $0 < \beta \ll 1$ and $\alpha \geq 1$

Here I consider an example with $0 < \beta < 1$, and to accentuate matters, with $\beta \ll 1$. In such theories matter and TM attract in the high acceleration limit but still repel in the deep-MOND limit. Since we have to have $\zeta < 1$ we take $\alpha \geq 1$, or, for concreteness’ sake take $\alpha = 1$.

From eq.(16) we can now write the potentials for the two sectors as

$$\phi = \phi^N + \beta^{-1}\hat{\phi}^N + \bar{\phi}, \quad \hat{\phi} = \phi^N + \beta^{-1}\hat{\phi}^N - \lambda\bar{\phi}, \quad (49)$$

where I used the fact that $\lambda = \beta/\alpha \ll 1$, and $\bar{\phi}$ is a solution of the second of eq.(14), i.e., $\vec{\nabla} \cdot \{\tilde{\mathcal{M}}'[(\vec{\nabla}\bar{\phi}/a_0)^2]\vec{\nabla}\bar{\phi}\} = 4\pi G(\rho - \hat{\rho})$. The gravitational potential that matter sees is thus the sum of three terms: its own Newtonian potential, the Newtonian potential of the TM enhanced by the large factor β^{-1} , and the “MOND” potential $\bar{\phi}$. Note that because here $\zeta \approx 1$, we have $\tilde{\mathcal{M}}'(\infty) \approx \lambda^{-1} \gg 1$, so for $a_0 \rightarrow 0$, $\bar{\phi} \rightarrow \lambda(\phi^N - \hat{\phi}^N)$.

We see that even a small admixture of TM in an otherwise pure matter object can make an important contribution to the matter potential, because its Newtonian potential is enhanced by the factor β^{-1} . For example, in such a theory, a small amount of TM trapped in the cores of (matter) galaxy clusters could explain away the mass discrepancies observed there. Such amounts of TM need not affect much the behavior in the MOND regime of the clusters, since TM enters the source of the second of eq.(14) with the same weight as matter.

It remains to be seen whether such asymmetric theories can be made consistent with cosmology. Recall that it is G/β that plays the role of the Newton constant in the BIMOND cosmologies I have considered so far. This can be interpreted as an apparent enhancement of the matter density by a factor $1/\beta$. For example, in the Friedmann equations a baryon density ρ_b appears as ρ_b/β , which would be interpreted as baryons plus DM of density $\rho_b(\beta^{-1} - 1)$.

6. Summary and discussion

I have studied some aspects of the dynamics of matter and the putative TM that may be present in the context of BIMOND. This is done for NR bodies of either type. In fully symmetric theories, which seem preferable on various grounds, we get MOND dynamics within each sector. The interaction between matter and TM is, however, nonstandard even compared with MOND: In the deep-MOND regime matter and TM bodies repel each other with MOND-like forces (decreasing as inverse distance). In the Newtonian, high-acceleration regime the force depends on the parameter β . For the fiducial value $\beta = 1$, the matter-TM interaction vanishes. For $\beta < 1$ there is Newtonian-like attraction (decreasing as inverse squared distance) with strength $\beta^{-1} - 1$, and for $\beta > 1$ there is similar repulsion.

I have also considered briefly possible effects of the presence of TM with such properties on structure formation and its lensing properties. To assess more reliably such effects we have to call upon numerical simulations.

It needs to be stressed, finally, that the TM does not, indeed cannot, play the full role of dark matter in galactic systems (although, as we saw, it may produce some effects attributed to dark matter): There isn't enough of it; it probably shies matter galactic systems; and, in all probability, would, anyhow, decrease gravitational attraction between matter bodies if it comes between them. It is still the MOND departure from standard gravity that is responsible for the observed mass discrepancy.

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